Two-degree-of-freedom Hamiltonian for the time-symmetric two-body problem of the relativistic action-at-a-distance electrodynamics

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We find a two-degree-of-freedom Hamiltonian for the time-symmetric problem of straight line motion of two electrons in direct relativistic interaction. This time-symmetric dynamical system appeared 100 years ago and it was popularized in the 1940s by the work of Wheeler and Feynman in electrodynamics, which was left incomplete due to the lack of a Hamiltonian description. The form of our Hamiltonian is such that the action of a Lorentz transformation is explicitly described by a canonical transformation (with rescaling of the evolution parameter). The method is closed and defines the Hamitonian in implicit form without power expansions. We outline the method with an emphasis on the physics of this complex conservative dynamical system. The Hamiltonian orbits are calculated numerically at low energies using a self-consistent steepest-descent method (a stable numerical method that chooses only the nonrunaway solution). The two-degree-of-freedom Hamiltonian suggests a simple prescription for the canonical quantization of the relativistic two-body problem.

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I. INTRODUCTION

The class of equivariant dynamical systems under the Poincaré group has enormous relevance to physics and yet, to date, only the one-body relativistic motion, is fully understood. Already with two bodies in relativistic motion, one encounters the no-interaction theorem: a group theoretical obstacle to the Hamiltonian description of relativistic two-particle motion [1]. The no-interaction theorem can be overcome by covariant constraint dynamics [2], but one is left with the few cases where the constraint scheme closes. For example, for an equivariant physical theory like the time-symmetric electrodynamics [3], a constraint description is unknown. In this paper we present a reduction of the time-symmetric two-body problem of the relativistic action-at-a-distance electrodynamics to a two-degree-of-freedom Hamiltonian system along the nonrunaway solutions. The form of the Hamiltonian is such that a Lorentz transformation is explicitly described by a canonical transformation with rescaling of the evolution parameter. The Hamiltonian orbits are calculated numerically by a numerically stable self-consistent method that uses steepest-descent quenching and chooses only the nonrunaway orbits.

In 1903, Schwarzchild proposed a relativistic type of interaction between charges that was time reversible precisely because it involved retarded and advanced interactions symmetrically [4]. The same model reappeared in the 1920s in the work of Tetrode and Fokker [5] and it finally became an interesting physical theory after Wheeler and Feynman showed that this direct-interaction theory can describe all the classical electromagnetic phenomena (i.e., the classical laws of Coulomb, Faraday, Ampère, and Biot-Savart) [3,6]. Another accomplishment was that Wheeler and Feynman showed in 1945 that in a certain limit where the electron practically interacts with a completely absorbing universe, the response of this universe to the electron's field is equivalent to the *local* Lorentz-Dirac self-interaction theory [7] without the need of mass renormalization [3]. It is amusing to understand that the classical radiative phenomena of Maxwell's electrodynamics can be described as a limiting case of this direct-interaction theory (complete absorption is added to the theory as a simple model to uncouple it from the detailed neutral-delay dynamics of the other charges of the universe; for other limits see Ref. [8]).

For the relativistic two-body system of the action-at-a-distance electrodynamics, general solutions are not known and the only known special solution is the circular orbit for the attractive two-body problem, first found in Ref. [9] and later rediscovered in Ref. [10] (see also Ref. [11]). Our problem has already been studied: the symmetric motion of two electrons along a straight line $[-x_2(t)=x_1(t)\equiv x(t)]$, which has the following equation in the action-at-a-distance electrodynamics:

$$m\frac{d}{dt}\left(\frac{v}{\sqrt{1-(v/c)^2}}\right)=\frac{e^2}{2r^2}\left(\frac{1-v(t-r)/c}{1+v(t-r)/c}\right)+\frac{e^2}{2q^2}\left(\frac{1+v(t+q)/c}{1-v(t+q)/c}\right), \quad (1)$$

where $v(t)\equiv dx/dt$ is the velocity of the first electron, of mass m and charge e, and r and q are the time-dependent delay and advance, respectively, which are implicitly defined by the light-cone conditions

$$cr(t)=x(t)+x(t-r), \quad (2)$$

$$cq(t)=x(t)+x(t+q),$$

where c is the speed of light. In general, a neutral-delay equation such as Eqs. (1) and (2) requires an initial function as the initial condition, but for the special case of Eqs. (1)

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and (2) it was proved in 1979 that for sufficiently low energies, the Newtonian initial condition [$x(0)=x_o$ and $v(0)=v_o$] determines a unique symmetric solution that is globally defined (i.e., that does not run-away at some point) [12,13]. This surprising uniqueness theorem reducing the initial condition from an arbitrary function to two simple real numbers (initial position and velocity) already suggests that the physical phase space could be isomorphic to a two-degree-of-freedom Hamiltonian vector field, at least for low velocities (which is what we find here). The first numerical method to solve Eqs. (1) and (2) was given in Ref. [14] and converged to solutions up to $v/c=0.94$. Later, another method [15] converged up to $v/c=0.99$.

In the following, we present a method to find the nonrunaway solution of Eqs. (1) and (2) with a two-degree-of-freedom Hamiltonian system. Our method is based on the physics; it starts from the Fokker action and transforms the neutral-advance-delay equation into two separate *Hamiltonian* ordinary differential equations for the same trajectory in two different foliations. The Hamiltonians are defined in implicit form and can be solved explicitly in terms of the arbitrary ghost functions by using the Hamilton-Jacobi theory. The condition that the two solutions describe the same trajectory poses a functional problem involving one of the ghost functions, with known asymptotic form, which must be solved self-consistently and is the basis of our numerical calculation of the Hamiltonian orbits. We outline the method with an emphasis on the physics described by this complex conservative dynamical system. The paper is organized as follows. In Sec. II we describe the bi-Lagrangian method and solve explicitly for the motion resulting from the Hamiltonians. In Sec. III we discuss some consequences of symmetry on the explicit solution of Sec. II to reduce the number of arbitrary functions. In Sec. IV we determine an equation to match the dynamics of one of the particles in the two foliations, which turns out to involve only one of the ghost functions. In Sec. V we use steepest-descent quenching to find the self-consistent Hamiltonian orbits. In Appendix A we prove the twice-monotonic property for the arbitrary-mass case. Appendix B discusses an alternative covariant derivation of the equal-mass case, and we also show here that the action of a Lorentz transformation on the Hamiltonian is represented by a canonical transformation with rescaling of the evolution parameter. In Sec. VI we give the conclusions and discussion.

II. OUTLINE OF THE METHOD

Here we consider the isolated two-body system with repulsive interaction, away from the other charges of the universe, a conservative time-reversible dynamical system in the action-at-a-distance electrodynamics. The equations of motion for two bodies in the action-at-a-distance electrodynamics of Wheeler and Feynman [3], henceforth called 1D-WF2B, Eqs. (1) and (2), are derived formally [16] by extremizing the Schwarzschild-Tetrode-Fokker action functional

$$S_F=-\int m_1 ds_1-\int m_2 ds_2 - e^2\int\int \delta(||x_1-x_2||^2)\dot{x}_1\cdot\dot{x}_2 ds_1 ds_2, \tag{3}$$

where $x_i(s_i)$ represents the four-position of particle $i=1,2$ parametrized by its arc length s_i, double bars stand for the four-vector modulus $||x_1-x_2||^2\equiv(x_1-x_2)\cdot(x_1-x_2)$, and the dot indicates the Minkowski scalar product of four-vectors with the metric tensor $g_{\mu\nu}$ ($g_{00}=1, g_{11}=g_{22}=g_{33}=-1$). The particles have masses m_1, m_2, common charge e, and in our units, $c=1$ [17]. The formal conserved energy associated with the Poincaré invariance of the Fokker Lagrangian (3) is discussed in Refs. [3,17], a nonlocal expression involving an integral over a portion of the trajectory, which is not useful to the present work, even though we start from the same Lagrangian (3).

The starting point of our method is a transformation to new variables

$$\xi_1\equiv t_1-x_1,\quad \zeta_1\equiv t_1+x_1, \tag{4}$$
$$\xi_2\equiv t_2-x_2,\quad \zeta_2\equiv t_2+x_2.$$

As first noticed in Ref. [16], this transformation splits the action integral (3) into two separate local actions

$$S_F=\tfrac{1}{2}(S_a+S_b), \tag{5}$$

with

$$S_a=-\int m_1(d\xi_1 d\zeta_1)^{1/2}-\int m_2(d\xi_2 d\zeta_2)^{1/2} - e^2\int\int \frac{\delta(\zeta_1-\zeta_2)}{|\xi_1-\xi_2|}(d\xi_1 d\zeta_2+d\xi_2 d\zeta_1), \tag{6}$$

and

$$S_b=-\int m_1(d\xi_1 d\zeta_1)^{1/2}-\int m_2(d\xi_2 d\zeta_2)^{1/2} - e^2\int\int \frac{\delta(\xi_1-\xi_2)}{|\zeta_1-\zeta_2|}(d\xi_1 d\zeta_2+d\xi_2 d\zeta_1). \tag{7}$$

It should be noticed that the double integral of Eq. (3) is reduced, after integration of the δ function, to a single integral over the parameter of particle 1, with particle 2 contributing only at the advanced and retarded positions, this being precisely the reason for the nonlocality of the theory, as illustrated in Eq. (1). The usefulness of parametrization (4) is that it naturally breaks the double integral of Eq. (3) into two integrals, each involving a different δ function, and integration over each δ function couples particle 1 with particle 2 at *either* the advanced position [the double integral included in S_a of Eq. (6)] *or* at the retarded position [the double integral included in S_b of Eq. (7)]. For example, in action S_a of Eq. (6), the nonzero contribution of the δ function occurs where the parameters ζ_1 and ζ_2 take equal values, $\zeta_1=\zeta_2=\zeta$, and this ζ is the natural independent parameter of the local action

S_a (this parametrization is often named the front form of dynamics in the literature and we henceforth call it type-*a* foliation). For action S_b of Eq. (7), integration over the δ function produces a contribution to the integral only where the two parameters ξ_1 and ξ_2 are equal, and $\xi_1=\xi_2=\xi$ is the natural time parameter of action S_b (henceforth called type-*b* foliation). To gain some insight into the two types of foliation, we notice that with type *a*, the particles are automatically in light-cone condition $(x_1-x_2)^2-(t_1-t_2)^2=0$, particle 2 always being ahead of particle 1 in time after the choice $x_1-x_2>0$, with the light-cone distance being

$$r_a=-\tfrac{1}{2}(\xi_1-\xi_2). \tag{8}$$

With type *b* parametrization, the particles are also in the light-cone condition, with particle 2 behind in time and the light-cone distance being

$$r_b=\tfrac{1}{2}(\zeta_1-\zeta_2). \tag{9}$$

The first heuristic guide for this work, as first noticed in Ref. [16], is the simplicity of the Euler-Lagrange problem for partial action (6): after expressing action (6) in terms of the timelike parameter ζ, it is easily verified that the associated Euler-Lagrange equation is a simple ordinary differential equation (not a delay equation anymore). The Euler-Lagrange problem for action (7) is analogous, with ζ replaced by ξ. To avoid confusion, we henceforth define that a Lagrangian has a *local* form when the associated Euler-Lagrange problem is defined by an ordinary differential equation. In the search for a local Lagrangian problem, we could try to extremize each of the partial action functionals of Eqs. (6) and (7) and obtain a trajectory by solving the Euler-Lagrange equation for either $\delta S_a=0$ or $\delta S_b=0$. Each separate minimization, in general, yields a different trajectory, which is the time-asymmetric problem studied in several works [18]. The main idea of our method is that *if* these two trajectories turn out to be equal, this common trajectory also extremizes the original action integral (3), as $\delta S_F=\frac{1}{2}\delta S_a+\frac{1}{2}\delta S_b=0+0=0$. Simply formulated as above, the problem turns out to be impossible; and it is possible to prove that the two separate solutions can never describe the same orbit. To overcome this difficulty we need to postulate a more general bi-Lagrangian problem by simultaneously solving

$$\delta S_a=\delta G \tag{10}$$

and

$$\delta S_b=-\delta G, \tag{11}$$

with G being a so far undetermined Lagrangian. A trajectory that satisfies Eq. (10) and (11) will also extremize the Fokker action (3), a simple consequence of Eqs. (5), (10), and (11):

$$\delta S_F=\tfrac{1}{2}\delta S_a+\tfrac{1}{2}\delta S_b=\tfrac{1}{2}\delta G-\tfrac{1}{2}\delta G=0. \tag{12}$$

Our first task is to find a sufficiently general Lagrangian G such that Eqs. (10) and (11) yield the same trajectory. Once we are trying to avoid delay equations, it is desirable that the Euler-Lagrange equations for Eqs. (10) and (11) be ordinary differential equations, which is the heuristic guide for choosing the functional G. A functional G that leaves the two separate problems (10) and (11) in local form is henceforth called a bilocal ghost Lagrangian G. Here we consider symmetric and time-reversible solutions of Eq. (1) only, but for the variational calculus that follows, it is necessary to study such an orbit immersed in a family of orbits, defined as follows: A time-reversible orbit naturally defines a preferred frame, the Lorentz frame where the orbit is time-reversible, and we henceforth call it the center of mass frame (CMF). We consider in the CMF the family of all orbits such that the trajectories of electrons 1 and 2 are both time reversible but not necessarily equal (nonsymmetric orbits) $[x_1(-t)=x_1(t)]$ and $[x_2(-t)=x_2(t)]$, and with the physical property that both the advanced and retarded distances decrease monotonically to a point of minimum and then start increasing monotonically again, as illustrated in Fig. 1. We henceforth call this family of orbits the CMF family. The fact that the solution of Eqs. (1) and (2) has this piecewise monotonic property is a consequence of the velocity being a monotonic function of time, which was proved in Ref. [12] for sufficiently low velocity orbits (in Appendix A we prove this assertion for the arbitrary-mass case). We henceforth refer to a CMF orbit as a twice-monotonic orbit. Since the solution we are looking for is symmetric and time reversible, it obviously belongs to the CMF family, and since this solution extremizes Eq. (3) in the family of all orbits, it obviously does so restricted to the CMF family. A symmetric and time-

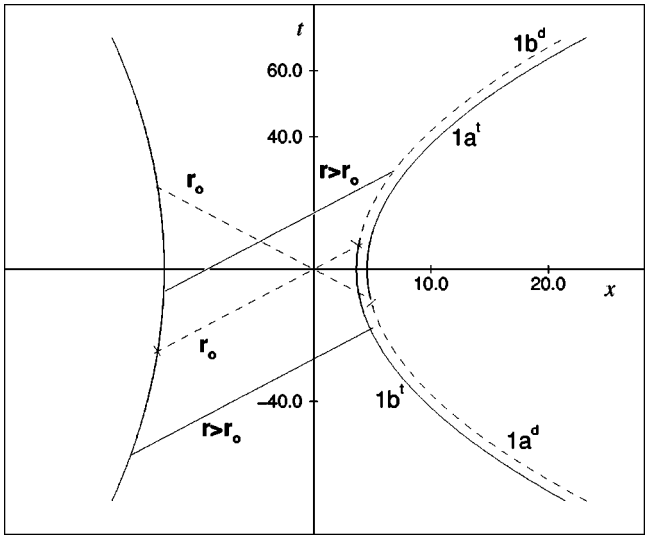

FIG. 1. Particle trajectories in the CMF for $m_1\neq m_2$ in the (x,t) plane, arbitrary units with $c=1$ and $e^2=mc^2$. Particle 1: case *a* indicated in branches: $1a^t$ (solid inner line on right) and $1a^d$ (hatched inner line on right); case *b* indicated in branches $1b^d$ (hatched outer line on right) and $1b^t$ (solid outer line on right). For clarity we indicate the type *a* and type *b* orbits of particle 1 as separate curves, but there is just one orbit for particle 1. Trajectory of particle 2 is represented by the solid line on left, branches are not indicated. The geometric distance in the (x,t) plane between two points in the light-cone condition is $\sqrt{2}r$, with r being the spatial distance, we have dropped the $\sqrt{2}$ factor and indicated simply r.

reversible orbit seen in a Lorentz frame other than the CMF has the property that the future of electron 1 is the past of electron 2 and vice versa. In the following we restrict the analysis of the different-mass case to the CMF. For a general covariant derivation of the equal-mass case, see Appendix B.

In the following we prove four integral identities for the orbits of the CMF family, which are later used to construct the bilocal ghost Lagrangian. The action of the time-reversal operation on orbits of the CMF family can be shown to be the following map: $\zeta_{1,2}\rightarrow-\xi_{1,2}, \xi_{1,2}\rightarrow-\zeta_{1,2}$, $r_a\rightarrow r_b$ and it is worth noticing that time-reversal maps type a parametrization onto type b and vice versa. In this work we ignore questions of convergence and define all integrals formally from $-\infty$ to ∞, an ambiguity inherited from the Wheeler-Feynman theory and discussed in Ref. [16]. The simplest type of integral identity we shall use, valid for an arbitrary function $\phi(x)$ of the real variable, is

$$\int_a \phi(\zeta)d\zeta=\int_b \phi(\xi)d\xi. \tag{13}$$

The lower index of the integral denotes the parametrization type, and the above identity is trivial, as with either type a or type b, the parameter runs from $-\infty$ to ∞ (ζ for type a and ξ for type b). It is also interesting to look at Eq. (13) as a consequence of the coordinate transformation induced by the time-reversal symmetry of the CMF family ($\zeta\rightarrow-\xi$). In the same way, we can prove in the CMF the following integral identity, involving an arbitrary function $V(x)$ of the real variable:

$$\int_a V(\zeta)\left(\frac{d\xi_1}{d\zeta}+\frac{d\xi_2}{d\zeta}\right)d\zeta=\int_b V(\xi)\left(\frac{d\zeta_1}{d\xi}+\frac{d\zeta_2}{d\xi}\right)d\xi. \tag{14}$$

The combination $(d\xi_1 d\zeta_2+d\xi_2 d\zeta_1)$ is the time-reversible Lorentz-invariant area element that appeared naturally in Eqs. (6) and (7). Last, the same time-reversal action ($\zeta_{1,2}\rightarrow-\xi_{1,2}, \xi_{1,2}\rightarrow-\zeta_{1,2}$) on the CMF family produces the following identities for arbitrary functions $\alpha(\zeta)$ and $\beta(\zeta)$ of the real variable:

$$\int_a \alpha_{d,t}(\zeta)(d\xi_1 d\zeta_1)^{1/2}=\int_b \alpha_{d,t}(\xi)(d\xi_1 d\zeta_1)^{1/2}, \tag{15}$$

$$\int_a \beta_{d,t}(\zeta)(d\xi_2 d\zeta_2)^{1/2}=\int_b \beta_{d,t}(\xi)(d\xi_2 d\zeta_2)^{1/2}.$$

The above identities suggest that we use a ghost Lagrangian G of type

$$G=\int_a\left[\phi(\zeta)+\frac{1}{2}V(\zeta)(\dot\xi_1+\dot\xi_2)+\alpha(\zeta)\sqrt{\dot\xi_1}+\beta(\zeta)\sqrt{\dot\xi_2}\right]d\zeta. \tag{16}$$

Notice that the dot over $\xi_{1,2}$ in Eq. (16) indicates the derivative with respect to ζ (the time parameter of case a). This G is in the local form when added to S_a, where ζ plays the role of the time parameter and the coordinates are ξ_1, ξ_2. When this same G is subtracted from action S_b, the integral identities allow us to express G as

$$G=\int_b\left[\phi(\xi)+\frac{1}{2}V(\xi)(\dot\zeta_1+\dot\zeta_2)+\alpha(\xi)\sqrt{\dot\zeta_1}+\beta(\xi)\sqrt{\dot\zeta_2}\right]d\xi, \tag{17}$$

which is also in the local form for action S_b, with ξ being the time parameter and the coordinates being ζ_1 and ζ_2. Notice that the dot over $\zeta_{1,2}$ in Eq. (17) indicates the derivative with respect to ξ (the time parameter of case b). One could, in principle, add more general parametrization-invariant terms to G: for example, terms involving the integration element $(d\xi_1 d\zeta_1)^{1/4}(d\xi_2 d\zeta_2)^{1/4}$ or any highly composite term, and the inversion to the Hamiltonian formalism would involve several branches. Lagrangian (16) is the most general ghost Lagrangian whose associated Hamiltonian involves quadratic rational functions of the momenta, and should suffice if the orbit has only two monotonic branches, corresponding to the two elements of the Galois group of a quadratic equation. The need for only four functions becomes also clearer later on, when we find that there are four determining equations involving these four arbitrary functions. We notice also that ϕ is defined up to a constant in Eqs. (16) and (17), which is also true of V, as adding a constant to V simply adds a total time derivative to G (a gauge transformation). There is also no gain in generality if one defines a general linear term like $V_1\dot\zeta_1+V_2\dot\zeta_2$ in Eq. (17), as this is also a trivial transformation of the case we used.

In the following we guide the reader to a division of the phase space into two disjoint regions, as our constructive method defines one Hamiltonian for each separate region as an implicit function of phase space: The condition $\dot r=0$ divides the phase space of a twice-monotonic orbit into two separate regions according to whether $\dot r>0$ or $\dot r<0$ (in Appendix B we show that this splitting is actually a covariant splitting for the equal-mass case). The change from ζ to r_a is one to one in each of the two regions of a twice-monotonic orbit $\zeta\in[-\infty,-|\zeta_c|]$ and $\zeta\in[-|\zeta_c|,\infty]$, as can be seen from Fig. 1, and this naturally splits all integrals into two. For example, the left integral of identity (13) splits as follows:

$$\int_a \phi(\zeta)d\zeta=\int_{-\infty}^{-|\zeta_c|}\phi_d(r_a)d\zeta+\int_{-|\zeta_c|}^{\infty}\phi_t(r_a)d\zeta. \tag{18}$$

The above integral identity involves two arbitrary functions, and the subscript t (as in turn) indicates that function ϕ_t is defined in the region of phase space where the trajectory of particle 1 includes a turning point (see Fig. 1), while subscript d (as in direct) indicates that ϕ_d is defined in the region of phase space where the trajectory of particle 1 is without a turning point (see Fig. 1). The same integral can be expressed for type b applying the time-reversal change of variable $\zeta\rightarrow-\xi$ to the right side of Eq. (18), which maps r_a

to r_b and maps the critical point $\zeta=-|\zeta_c|$ of the ζ parametrization to the critical point $\xi=|\zeta_c|$ of the ξ parametrization,

$$\int_b \phi(\xi)d\xi=\int_{-\infty}^{|\zeta_c|}\phi_t(r_b)d\xi+\int_{-|\zeta_c|}^{\infty}\phi_d(r_b)d\xi. \quad (19)$$

These two portions are indicated in Fig. 1 for both case *a* and case *b*. In the following we split all integrals of the ghost Lagrangians (16) and (17) into two, which we indicate with subscripts *t* and *d* in the same way as Eqs. (18) and (19). The usefulness of the above splitting of the phase space is that one can express all functions in the ghost Lagrangian as functions of the light-cone distance (8) in each region, such that the Lagrangian becomes independent of the timelike parameter and allows the existence of a conserved energy *E*.

Our next task is to solve each separate problem for each separate Lagrangian (and for each region of the phase space). After inclusion of the ghost Lagrangian, the problem $\delta S_a=\delta G$ in the CMF family implies the Euler-Lagrange equations for $L_a=S_a-G$,

$$L_a^{t,d}=-\int\left[M_{1a}^{t,d}\sqrt{\dot{\xi}_1}+M_{2a}^{t,d}\sqrt{\dot{\xi}_2}+\left(\frac{e^2}{|\xi_1-\xi_2|}+\frac{1}{2}V_{t,d}(r_a)\right)\times(\dot{\xi}_1+\dot{\xi}_2)+\phi_{t,d}(r_a)\right]d\zeta, \quad (20)$$

where $M_{1a}^{t,d}\equiv m_1+\alpha_{t,d}(r_a)$ and $M_{2a}^{t,d}\equiv m_2+\beta_{t,d}(r_a)$ and the Lagrangian can be uniquely inverted in each branch to produce a Hamiltonian, because of the monotonic property. The problem $\delta S_b=-\delta G$ is described by $L_b=S_b+G$,

$$L_b^{t,d}=-\int\left[M_{1b}^{t,d}\sqrt{\dot{\zeta}_1}+M_{2b}^{t,d}\sqrt{\dot{\zeta}_2}+\left(\frac{e^2}{|\zeta_1-\zeta_2|}-\frac{1}{2}V_{t,d}(r_b)\right)\times(\dot{\zeta}_1+\dot{\zeta}_2)-\phi_{t,d}(r_b)\right]d\xi, \quad (21)$$

with $M_{1b}^{t,d}\equiv m_1-\alpha_{t,d}(r_b)$ and $M_{2b}^{t,d}\equiv m_2-\beta_{t,d}(r_b)$. We have introduced eight arbitrary ghost functions: $\phi_{t,d}$, $V_{t,d}$, $\alpha_{t,d}$ and $\beta_{t,d}$, four for each separate region of phase space, and we notice that these ghost functions enter with a plus sign in case *a* and with a minus sign in case *b*. The Hamiltonian in each case is given by

$$H_a=\frac{-1}{4}\left\{\frac{M_{1a}^2}{\left(p_1+\frac{1}{2}V+\frac{e^2}{|\xi_1-\xi_2|}\right)}+\frac{M_{2a}^2}{\left(p_2+\frac{1}{2}V+\frac{e^2}{|\xi_1-\xi_2|}\right)}\right\}-\phi(r_a), \quad (22)$$

and

$$H_b=\frac{-1}{4}\left\{\frac{M_{1b}^2}{\left(p_1-\frac{1}{2}V+\frac{e^2}{|\zeta_1-\zeta_2|}\right)}+\frac{M_{2b}^2}{\left(p_2-\frac{1}{2}V+\frac{e^2}{|\zeta_1-\zeta_2|}\right)}\right\}+\phi(r_b). \quad (23)$$

We have omitted the subscripts but it should be kept in mind that each of the above Hamiltonians is defined separately in each region of the phase space, a separation that will be useful when we come to the symmetry considerations. Notice that the Hamiltonian H_a depends only on $r_a=-\frac{1}{2}(\xi_1-\xi_2)$, which implies that $P_a=p_1+p_2$ is a constant of motion. For type *b* parametrization, Hamiltonian H_b depends only on $r_b=\frac{1}{2}(\zeta_1-\zeta_2)$, implying the constant $P_b=p_1+p_2$. The constant $P_a=p_1+p_2$ suggests a canonical change of variables for Hamiltonian H_a, defined by

$$X\equiv\tfrac{1}{2}(\xi_1+\xi_2),\quad P\equiv p_1+p_2, \quad (24)$$

$$x\equiv\tfrac{1}{2}(\xi_1-\xi_2),\quad p=p_1-p_2.$$

For type *b* we use the analogous transformation with ξ replaced by ζ in the above formulas. One can use Eq. (24) to express p_1 and p_2 of Eq. (22) in terms of the constant $P=P_a$ and the relative momentum *p*, and substitution into the condition $H_a=E_a$ yields a quadratic equation for *p*, with solutions

$$p_a=\frac{\Delta_a}{(E_a+\phi)}\pm\sqrt{\left(P_a+\frac{Q_a}{(E_a+\phi)}+V(r)+\frac{e^2}{r_a}\right)^2+\left(\frac{\Delta_a^2-Q_a^2}{(E_a+\phi)^2}\right)}, \quad (25)$$

where $Q_a\equiv\frac{1}{4}(M_{1a}^2+M_{2a}^2)$ and $\Delta_a\equiv\frac{1}{4}(M_{2a}^2-M_{1a}^2)$ and $r_a=|x|$. The separation for case *b* is analogous.

So far we have shown that any common solution of Hamiltonians (22) and (23) is also a solution of the original advance-delay problem of Eqs. (1) and (2), for arbitrarily given potentials $\phi_{t,d}, V_{t,d}, \alpha_{t,d}$, and $\beta_{t,d}$. It turns out that, even if we guessed the four potentials correctly, Hamiltonians (22) and (23) would have only a single trajectory in common for each given set of potentials (this becomes clear in the numerical work of Sec. V). This obstacle can be overcome with the Hamiltonian formalism if we generalize the potentials of Eqs. (22) and (23) to implicit functions of the energy $E_a=H_a$ in case *a* and of $E_b=H_b$ in case *b*. For example, the potential ϕ is generalized to $\phi=\phi(r_a,E_a)$ in case *a* and to $\phi=\phi(r_b,E_b)$ in case *b* (an analogous generalization goes for *V*, α, and β). This generalized Hamiltonian is still a function of phase space, because *E* itself is a function of phase space, even though it is now only implicitly defined by Eqs. (22) and (23). In this generalization, for each given orbit, of energy E_o, we still define the ghost

Lagrangians with Eqs. (20) and (21) using *fixed form* potentials: $\phi=\phi(r,E_o)$, $V=V(r,E_o)$, $\alpha=\alpha(r,E_o)$, and $\beta=\beta(r,E_o)$. By construction, these generalized ghost Lagrangians have only a single orbit in common with the generalized Hamiltonians, but it is essential that such provisional Lagrangians exist, such that we can prove that the Hamiltonian equations associated to Eqs. (22) and (23) lead to Eq. (1), which is accomplished by use of Eq. (12) with fixed form potentials. After that we can dispose of the Lagrangians. On the Hamiltonian side, if we are changing the potentials with E, the Hamiltonian equations of motion derived from Eqs. (22) and (23) pick an extra term proportional to the derivative of the Hamiltonian with respect to E, (due to the implicit dependence). We must therefore supplement a condition that this derivative vanishes along the orbit in each case, which in case a reads

$$\frac{\partial H_a(p,P,r,E_a)}{\partial E_a}=0 \tag{26}$$

and in case b reads

$$\frac{\partial H_b(p,P,r,E_b)}{\partial E_b}=0. \tag{27}$$

In Eq. (26), the derivative $\partial H_a/\partial E_a=0$ should hold on the energy shell $H_a=E_a$, and in Eq. (27), the derivative $\partial H_b/\partial E_b=0$ should hold on the energy shell $H_b=E_b$. Elimination of the relative momentum p_a from $H_a=E$ yields Eq. (25), and substitution of Eq. (25) into Eq. (26) yields a partial differential equation (PDE) involving the four potentials. An analogous PDE results for Eq. (27) in case b such that Eqs. (26) and (27) define two partial differential equations involving the four arbitrary potentials in each region (variables of the partial differential equations are r, P, and E). Rigorously, the generalization to implicitly defined Hamiltonians proceeds only if the time-reversal operation also maps E_a into E_b. For that we notice that ϕ enters with a plus sign in H_a [Eq. (22)] and with a minus sign in H_b [Eq. (23)], and the required symmetry can be accomplished by adding an energy dependent constant to ϕ. We conclude this paragraph stressing that the generalized ghost Lagrangians were only a provisional artifact *en route* to the eventual derivation of the Hamiltonians (22) and (23) from a variational argument with use of symmetry. It should be clear that after we generalize Eqs. (22) and (23) to implicit dependence and postulate Eqs. (26) and (27), we can no longer go back to the simple provisional ghost Lagrangians, and our constructive approach is *essentially* left with an implicitly defined bi-Hamiltonian system.

In the following we show that even with Hamiltonian (22) defined in the implicit form we can write out the motion explicitly: This explicit solution is accomplished in the manner of Hamilton-Jacobi, by use of a canonical transformation with a generating function S given by

$$S=PX+W(x,P,E)-E\zeta, \tag{28}$$

where the function $W(x,P,E)$ is defined by integration from the condition $p=\partial W/\partial x$, with p given by Eq. (25). This canonical transformation is defined such that the new momentum associated with the old variable X is the same old constant $P=\partial S/\partial X$ and the other new momentum is the energy E (with this last definition we exploit the fact that E is already one argument of the potentials). We choose S in the manner of Hamilton-Jacobi such that the new Hamiltonian vanishes: $K=H+\partial S/\partial\zeta=0$. As the Hamiltonian is zero, the new coordinates are defined simply by two constants X_0 and C_0:

$$X_0=\partial S/\partial P=X+\partial W/\partial P, \tag{29}$$

$$C_0=-\partial S/\partial E=\zeta-\partial W/\partial E.$$

The above equations for type a define ζ and X as functions of the variable $r_a\equiv|x|$, and provide the complete solution of the Hamiltonian motion. For further use, it is interesting to take the differentials of Eq. (29) relative to x,

$$dX=-(\partial^2 W/\partial x\partial P)dx=-(\partial p/\partial P)dx, \tag{30}$$

$$d\zeta=(\partial W/\partial x\partial E)dx=(\partial p/\partial E)dx,$$

where we have used $p=\partial W/\partial x$ (definition of the Hamilton-Jacobi transformation) and exchanged the partial derivatives. The explicit form of the differential for the trajectory is obtained using Eq. (4) to relate particle coordinates to X and ζ and using (30) to relate dX and $d\zeta$ to dx. For type a parametrization the explicit solution is

$$dt_{1a}=\frac{1}{2}(d\zeta_a+dX_a+dx_a)=\frac{1}{2}\left(\frac{\partial p_a}{\partial P}-\frac{\partial p_a}{\partial E}-1\right)dr_a, \tag{31}$$

$$dt_{2a}=\frac{1}{2}(d\zeta_a+dX_a-dx_a)=\frac{1}{2}\left(\frac{\partial p_a}{\partial P}-\frac{\partial p_a}{\partial E}+1\right)dr_a,$$

$$dx_{1a}=\frac{1}{2}(d\zeta_a-dX_a-dx_a)=\frac{1}{2}\left(1-\frac{\partial p_a}{\partial P}-\frac{\partial p_a}{\partial E}\right)dr_a,$$

$$dx_{2a}=\frac{1}{2}(d\zeta_a-dX_a+dx_a)=-\frac{1}{2}\left(\frac{\partial p_a}{\partial P}+\frac{\partial p_a}{\partial E}+1\right)dr_a,$$

where we have also used $dx_a=-dr_a$. Analogously for type b ($dx_b=dr_b$) we obtain the explicit solution

$$dt_{1b}=\frac{1}{2}(d\xi_b+dX_b+dx_b)=\frac{1}{2}\left(\frac{\partial p_b}{\partial E}-\frac{\partial p_b}{\partial P}+1\right)dr_b, \tag{32}$$

$$dt_{2b}=\frac{1}{2}(d\xi_b+dX_b-dx_b)=\frac{1}{2}\left(\frac{\partial p_b}{\partial E}-\frac{\partial p_b}{\partial P}-1\right)dr_b,$$

$$dx_{1b}=\frac{1}{2}(-d\xi_b+dX_b+dx_b)=\frac{1}{2}\left(1-\frac{\partial p_b}{\partial P}-\frac{\partial p_b}{\partial E}\right)dr_b,$$

$$dx_{2b}=\frac{1}{2}(-d\xi_b+dX_b-dx_b)=-\frac{1}{2}\left(\frac{\partial p_b}{\partial P}+\frac{\partial p_b}{\partial E}+1\right)dr_b.$$

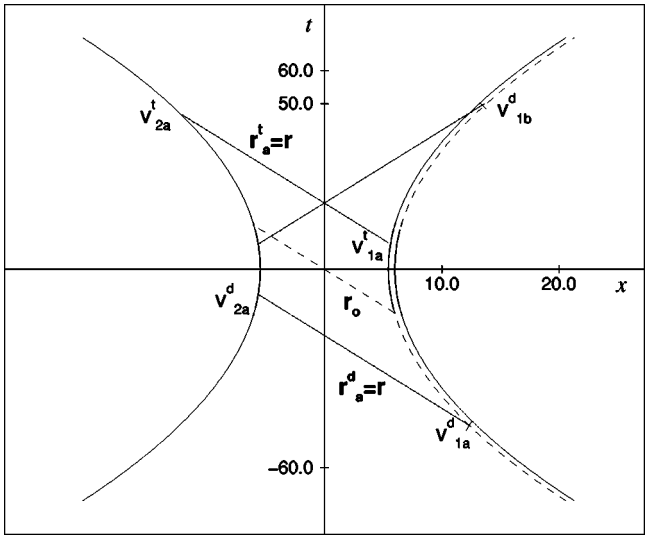

FIG. 2. Particle trajectories in the CMF for $m_1=m_2$ in the (x,t) plane, arbitrary units with $c=1$ and $e^2=mc^2$. Indicated is the minimum distance r_o and the velocity of particle 1 in each branch corresponding to the same distance $r=r_a^d=r_a^t>r_o$. Event v_{1a}^d is the time-reversed point of event v_{1b}^d and, because of that, $r_a^d=r_b^d=r$. A special symmetry of the equal-mass case: event v_{2a}^t is obtained by particle exchange and time reversal from v_{1a}^d, which implies $r_a^t=r_b^d=r$. All branches of the trajectory of particle 1 are indicated, and omitted for particle 2. Notice that we indicated the spatial distance in the light-cone, which is $\sqrt{2}$ times the geometric distance in the (x,t) plane.

We recall that Eqs. (31) and (32) give the explicit solution in terms of $p(r,E,P)$ as given by Eq. (25).

III. SYMMETRY CONDITIONS FOR THE EQUAL-MASS CASE

In this section we discuss only the equal-mass case, and for that we set $m_1=m_2=m=1$ and allow only the charge to be arbitrary, from which the general m and c case can be recovered by simply replacing e^2 by (e^2/mc^2). We henceforth set $e=1$ as well, which can be accomplished by a rescaling of distances accompanied by a rescaling of time to keep $c=1$. In the following we derive general symmetry relations involving the eight arbitrary functions $\alpha_{t,d}, \beta_{t,d}, V_{t,d}$, and $\phi_{t,d}$. Formula (25) for p_a is the solution of a quadratic equation and defines two different functions p_a by taking the plus and minus signs of the square root. It is easy to show that at the branch point the square root vanishes, so that a single branch of the square root describes each of the t and d physical regions of phase space as indicated in Fig. 1, henceforth indicated by p_a^t and p_a^d (on $p_{a,b}$ we use superscripts to indicate branch type, to avoid overloaded notation, but with the ghost functions we keep using subscripts). We have assumed that the orbit is time reversible in the CMF and, to be consistent with that, time reversal must map each branch of the type a trajectory of particle 1 onto a branch of its type b trajectory, with the corresponding velocities transforming like $v_{1a,b}^{d,t}(r)=-v_{1b,a}^{d,t}(r)$ for $r\in[r_o,\infty]$, as illustrated in Fig. 2. In an analogous way, for particle 2 we should have $v_{2a,b}^{d,t}(r)=-v_{2b,a}^{d,t}(r)$ for $r\in[r_o,\infty]$. These two symmetry conditions, when expressed in terms of $\partial p/\partial P$ and $\partial p/\partial E$ using Eqs. (31) and (32), imply the four conditions

$$\frac{\partial p_{a,b}^{t,d}}{\partial P}=\frac{\partial p_{b,a}^{t,d}}{\partial P}, \tag{33}$$

$$\frac{\partial p_{a,b}^{t,d}}{\partial E}=\frac{\partial p_{b,a}^{t,d}}{\partial E}. \tag{34}$$

Conditions (33) and (34) represent two conditions for region t and two conditions for region d, each involving the corresponding set of four potentials. For example, in region t condition (33) is a simple algebraic equation because the potentials do not depend on P explicitly (a possible physical choice on the CMF), while condition (34) is a partial differential equation. We have completed the determining equations for the potentials, which, for example, in region t is composed of Eqs. (26) and (27) together with the t sector of Eqs. (33) and (34). At this point we notice another reason to include only four ghost potentials, as we found four determining equations to be satisfied [Eqs. (26), (27), (33), and (34)]. The solution to these determining partial differential equations should determine the potentials in the CMF. This solution is elaborate and still involves arbitrary initial functions of r, which must be determined numerically, which we discuss elsewhere [19]. In this work we calculate the Hamiltonian orbits directly with an independent numerical method.

Finally we notice the following time-reversal-and-exchange symmetry relating region d of case a to region t of case b of the equal-mass case: As the direction of time in the CMF is arbitrary and the particles are identical, the Lagrangian for the d branch of case a must be equal to the Lagrangian for the t branch of case b with particles exchanged and vice versa, which implies

$$\alpha_t=-\beta_d, \quad \beta_t=-\alpha_d, \tag{35}$$

$$\phi_t=-\phi_d, \quad V_t=-V_d.$$

An immediate consequence of Eq. (35) is that $E_a^t=E_b^d$ and $E_a^d=E_b^t$ as well as $P_a^t=P_b^d$ and $P_a^d=P_b^t$. Because ϕ and V are arbitrarily defined up to gauge constants, and as ϕ and V enter with a plus sign in case a and with a minus sign in case b, we can also choose $E_a^t=E_b^t=E$ and $P_a^t=P_b^t=P$, such that one can use a common value for all the energies and a common value for all momenta, throughout the four combinations of region and case. We henceforth indicate energies simply by E and momenta by P.

IV. EQUATION OF MATCHING FOR THE EQUAL-MASS CASE

In this section we introduce a simpler description in terms of two simple functions $s(r,E)$ and $F(r,E)$ (that are immediately accessible numerically). For example, the d sector of Eqs. (33) and (34) is studied by defining $\partial p_a^d/\partial P$, $\partial p_a^d/\partial E$, $\partial p_b^d/\partial P$, and $\partial p_b^d/\partial E$ in terms of $s(r,E)$ and $F(r,E)$ as

$$\frac{\partial p_a^d}{\partial P}=\frac{\partial p_b^d}{\partial P}\equiv-\frac{\cosh[s(r,E)]}{\sinh[s(r,E)]}, \tag{36}$$

$$\frac{\partial p_a^d}{\partial E}=\frac{\partial p_b^d}{\partial E}\equiv\frac{F(r,E)}{\sinh[s(r,E)]}. \tag{37}$$

For branch *t*, the consequences of definitions (36) and (37) and the symmetry relations of Eq. (35) are (i) that branch *t* involves the same function $F(r,E)$ of branch *d* and (ii) that branch *t* involves the function $s(r,E)$ of branch *d* with a change of sign. The general picture is that $\pm s(r,E)$ and $F(r,E)$ describe both case *a* and case *b*, exchanging branches in the same case replaces $s(r,E)$ by $-s(r,E)$, while exchanging case for the same branch leaves functions $s(r,E)$ and $F(r,E)$ unchanged. In the following we drop the dependence on *E* of the functions for brevity.

Now we must impose that the same orbit is a solution of both Eqs. (22) and (23), which demands that the *d* portion of the *a* orbit of particle 1 should coincide with a *piece* of the *t* branch of particle 1 in case *b* (see Fig. 1). Notice that this is not the one-to-one branch correspondence of the symmetry considerations of Sec. III, and we stress the word *piece*, because the branches are changed at different points, as can be seen from Fig. 1. We shall henceforth drop the subscript notation, and simply write s_a and s_b, meaning the plus or the minus branch of the function $s(r)$, wherever it applies. We can use Eqs. (31), (32) and (36), (37) to express the differentials of the particle-1 coordinates with type *a* foliation in terms of r_a and $s_a=\pm s(r_a)$:

$$dt_{1a}+dx_{1a}=\frac{-1}{\sinh(s_a)}F(r_a)dr_a, \tag{38}$$

$$dt_{1a}-dx_{1a}=-\frac{\exp(s_a)}{\sinh(s_a)}dr_a,$$

and with type *b*,

$$dt_{1b}+dx_{1b}=\frac{\exp(s_b)}{\sinh(s_b)}dr_b, \tag{39}$$

$$dt_{1b}-dx_{1b}=\frac{1}{\sinh(s_b)}F(r_b)dr_b.$$

At this point it is convenient to introduce still another function: the velocity function of particle 1, which must be the same in the corresponding branches of each case. We define it in case *a* by $v_{1a}(s_a,r_a)=\tanh(\Phi)$ and in case *b* by $v_{1b}(s_b,r_b)\equiv\tanh(\Phi)$, which yields

$$\exp(2\Phi)\equiv\exp(-s_a)F(r_a)=\exp(s_b)F^{-1}(r_b). \tag{40}$$

The first condition of matching for the trajectory of particle 1 as described by the two foliations, requires that the velocities be the same, $v_{1a}(s_a,r_a)=v_{1b}(s_b,r_b)$, resulting in

$$\exp(s_a)\exp(s_b)=F(r_a)F(r_b), \tag{41}$$

a rearrangement of Eq. (40). It is important to stress that differently from the symmetry conditions, in this condition r_a and r_b are not equal, but rather for every pair (s_a,r_a) we should be able to find a pair (s_b,r_b) such that Eq. (41) is satisfied. Figure 2 illustrates yet another symmetry special for the equal-mass case: while particle 1 has a velocity angle $\Phi_t(r_a)$ (event v_{1a}^t in Fig. 2), particle 2 has a velocity angle of $\Phi_d(r_a)$ the same velocity particle 1 had in the past at the first time that $r_a=r$ [this symmetry reads $v_{2a}^t(r)=v_{1a}^d(r)$]. Notice that there are two points along the orbit where the advanced distance assumes a given value, one in the *t* branch where the velocity angle of particle 1 is $\Phi_t(r_a)$ and one in the *d* branch with velocity angle $\Phi_d(r_a)$, as illustrated in Fig. 2. With the understanding that these two branches must be produced with opposite signs for the function $s(r)$, Eq. (40) implies that

$$\exp(\Phi_d)\exp(\Phi_t)=F(r_a), \tag{42}$$

which in turn shows that $F(r)$ is determined by past data only, namely the function $\Phi_d(r_a)$. Another consequence of Eq. (42) is that $F(\infty)=1$, as $\Phi_d(\infty)=-\Phi_t(\infty)$, the asymptotic boundary condition on the CMF. Once the orbit is described by two differentials, there is another condition for the orbits to be parallel at all times, which is most easily expressed by equating the relativistic proper time of particle 1 in the two foliations:

$$(d\tau_1)^2=\frac{\exp(s_a)}{\sinh^2(s_a)}F(r_a)(dr_a)^2=\frac{\exp(s_b)}{\sinh^2(s_b)}F(r_b)(dr_b)^2. \tag{43}$$

From the above we can derive differential equations for the motion of r_a and r_b:

$$\frac{dr_a}{d\tau_1}=-\frac{\exp(-s_a/2)\sinh(s_a)}{\sqrt{F(r_a)}}, \tag{44}$$

$$\frac{dr_b}{d\tau_1}=\frac{\exp(-s_b/2)\sinh(s_b)}{\sqrt{F(r_b)}}.$$

Notice that we have used opposite signs for the evolution of r_a and r_b, the only sensible choice. Equation (44) describes a decrease of r_a and r_b at large distances if $s_a>0$ and $s_b<0$ (ingoing asymptotics) and an increase of r_a and r_b at large distances when $s_a<0$ and $s_b>0$ (outgoing asymptotics). While asymptotically s_a and s_b must have opposite signs, they do not change sign at the same point and, in particular, in the turning region of particle 1 they have the same sign, as illustrated in Fig. 1. We can also eliminate s_a and s_b in favor of Φ from Eq. (44), resulting in

$$\frac{dr_a}{d\tau_1}=-\frac{1}{2}\left[\exp(-\Phi)-\frac{\exp(3\Phi)}{F^2(r_a)}\right], \tag{45}$$

$$\frac{dr_b}{d\tau_1}=\frac{1}{2}\left[\exp(\Phi)-\frac{\exp(-3\Phi)}{F^2(r_b)}\right].$$

To close the dynamical system of matching we need an equation for the variable Φ; which is provided by the Wheeler-Feynman equation of motion (1). To obtain a local equation, we write Eq. (1) using a combination of type a and type b foliations in the following way: whenever we need the advanced position of particle 2, we write it using type a foliation (as particle 2 is naturally in the future light cone), while the retarded position of particle 2 is simply written with type b foliation (where particle 2 is naturally in the past light cone). The usefulness of the variable Φ is discovered when Eq. (1) is written in terms of Φ and the proper time of particle 1, which yields simply (recall that we are using $e=1$)

$$\frac{d\Phi}{d\tau_1}=\frac{1}{2}\left\{\frac{\exp(2\Phi)}{r_a^2F^2(r_a)}+\frac{\exp(-2\Phi)}{r_b^2F^2(r_b)}\right\}. \tag{46}$$

Equations (45) and (46) constitute the complete ordinary differential equation (ODE) to describe the matching for the orbit of particle 1. By now we have turned Eq. (1) upside down and used all the symmetries, and the resulting Eqs. (45) and (46) are much simpler to solve than Eq. (1). Rigorously speaking we now have a delay-only equation, as $F(r)$ depends only on past data via Eq. (42). This should be contrasted with Eq. (1), a neutral-delay-advance equation with infinite lags. To solve Eqs. (45) and (46) one needs to postulate an arbitrary positive function $F(r)$ with a given asymptotic form $F(\infty)=1$, and solve the resulting ODE. For self-consistency, the ghost function $F(r)$ must be chosen such that the orbit of particle 2 is the same even function as that of particle 1, the definition of CMF. This functional problem is solved numerically in the following section.

Last we illustrate how to express the potentials of the CMF in terms of the numerically accessible functions $s(r,E)$ and $F(r,E)$, along the simplest type of solution to Eqs. (26), (27), (36), and (37). As in the CMF, the potentials depend only on E, Eq. (36) is algebraic and can be solved simply in both case a and case b as

$$\left(P_a+V+\frac{e^2}{r}\right)=\frac{[2(1+\alpha)(1+\beta)\cosh s-(1+\alpha)^2-(1+\beta)^2]}{4(E_a+\phi)}, \tag{47}$$

$$\left(P_b-V+\frac{e^2}{r}\right)=\frac{[2(1-\alpha)(1-\beta)\cosh s-(1-\alpha)^2-(1-\beta)^2]}{4(E_b-\phi)}. \tag{48}$$

For case a the equations of motion for ξ_1 and ξ_2 derived from Eqs. (22) and (26) are

$$\frac{\exp(s)}{F(r)}=\frac{d\xi_1}{d\zeta}=\frac{(1+\alpha)^2}{4\left(p_1+\frac{1}{2}V+\frac{e^2}{2r}\right)^2}, \tag{49}$$

$$\frac{\exp(-s)}{F(r)}=\frac{d\xi_2}{d\zeta}=\frac{(1+\beta)^2}{4\left(p_2+\frac{1}{2}V+\frac{e^2}{2r}\right)^2}. \tag{50}$$

We now take the most physically sensible square root of Eqs. (49) and (50), and substitute into Eq. (22), yielding

$$E_a+\phi_d=\frac{1}{2\sqrt{F}}[(1+\alpha_d)\exp(s/2)-(1+\beta_d)\exp(-s/2)]. \tag{51}$$

In the same way, for case b, we obtain after a choice of sign for the square root

$$E_b-\phi_d=\frac{-1}{2\sqrt{F}}[(1-\alpha_d)\exp(s/2)-(1-\beta_d)\exp(-s/2)]. \tag{52}$$

As discussed below Eq. (35), we henceforth set $E_b^d=E_a^d\equiv E$ and $P_b^d=P_a^d\equiv P$. A linear equation involving α_d and β_d is found by adding Eqs. (51) and (52), yielding

$$\alpha_d\exp(s/2)-\beta_d\exp(-s/2)=2E\sqrt{F}. \tag{53}$$

Another linear equation involving α and β can be obtained by substituting Eq. (51) into Eq. (47), Eq. (52) into Eq. (48), and adding the resulting Eqs. (47) and (48), obtaining

$$\alpha_d\exp(-s/2)-\beta_d\exp(s/2)=-\frac{2\left(P+\frac{e^2}{r}\right)}{\sqrt{F}}, \tag{54}$$

and we notice that Eqs. (53) and (54) constitute two linear equations for α and β. At the shortest light-cone distance r_o, which happens at $s(r_o)=0$, the determinant of the linear system vanishes and poses the following solvability condition involving E and P:

$$E=-\frac{\left(P+\frac{e^2}{r_o}\right)}{F(r_o)}. \tag{55}$$

For $r>r_o$ one has $s(r)\neq 0$ and the linear system (53) and (54) can be solved for α_d and β_d yielding

$$\alpha_d=\frac{\left[EF(r)\exp(s/2)+\left(P+\frac{e^2}{r}\right)\exp(-s/2)\right]}{\sqrt{F(r)}\sinh(s)}, \tag{56}$$

$$\beta_d=\frac{\left[EF(r)\exp(-s/2)+\left(P+\frac{e^2}{r}\right)\exp(s/2)\right]}{\sqrt{F(r)}\sinh(s)}.$$

In the CMF, the constant P is a function of E, such that the potentials depend only on r and E, but the detailed analytical form is not trivial at this point. It is also easy to find ϕ_d and V_d,

$$\phi_d = \frac{\sinh(s/2)}{\sqrt{F}}, \tag{57}$$

$$V_d = \sqrt{F}\sinh(s/2).$$

It can be checked that the t branch solution can be obtained from Eqs. (56) and (57) by setting $s \rightarrow -s$, which is symmetry (35) for equal-mass particles in the CMF.

We warn the reader that this simplest type of solution to Eqs. (26), (27), (33), and (34) does not correspond to the low-energy orbits studied in the following section, and it was included only to illustrate how the potentials can be determined by $s(r,E)$ and $F(r,E)$. A complete study of all possible solutions is elaborate and will be published elsewhere.

V. NUMERICAL INTEGRATION: THE STEEPEST-DESCENT METHOD

In Sec. IV we saw that describing the same particle 1 in both foliations results in Eqs. (45) and (46), involving the single unknown ghost function $F(r)$. The above discussion suggests the following simple self-consistent method to obtain the symmetric solution of 1D-WF2B in the equal-mass case in the CMF: We start by postulating the functional form of $F(r)$, which must go to 1 at large distances, as noted below Eq. (42). For the following numerical work we use up to 18 arbitrary coefficients to approximate $F(r)$ by a truncated power series,

$$F(r) = 1 - \sum_{n=1}^{n=18} \frac{k_n}{r^n}, \tag{58}$$

which has the desired asymptotic form. After we assume given values for the k_n, the main equations (45) and (46) yield a simple initial value ODE problem. The integration can be carried out froms the turning point, where $\tau_1 = 0$, $\Phi = 0$, and $r_a = r_b = r_c > r_o$ (the given functional form of $F(r)$ and r_c determine all the subsequent dynamics). This integration produces a time-reversible orbit for particle 1. It can be seen by inspection of Eqs. (45) and (46) that exchanging τ_1 and $-\tau_1$, and Φ and $-\Phi$ simply exchanges r_a and r_b, such that $r_a(-\tau_1) = r_b(\tau_1)$, a consequence of the symmetry imposed. When we start particle 1 at the turning point, $\Phi = 0$, particle 2 is described in type a parametrization at the advanced point $(x_{2a}, t_{2a}) = (-r_c, r_c)$, while with type b parametrization particle 2 is at the retarded position $(x_{2a}, t_{2a}) = (-r_c, -r_c)$. As illustrated in Fig. 3, for a generic choice of $F(r)$ the future of type b trajectory of particle 2 will not match the type a trajectory of the same particle 2, which starts ahead of case b, and the scheme produces two different orbits for particle 2, which is absurd. It is necessary to adjust the function $F(r)$ precisely to obtain a single trajectory for particle 2, and it is nice to observe that the asymptotic condition $F(\infty) = 1$ guarantees the asymptotic velocity of particle 2 to be the same in both foliations, so if we adjust the orbits to overlap in the turning region, they become close everywhere.

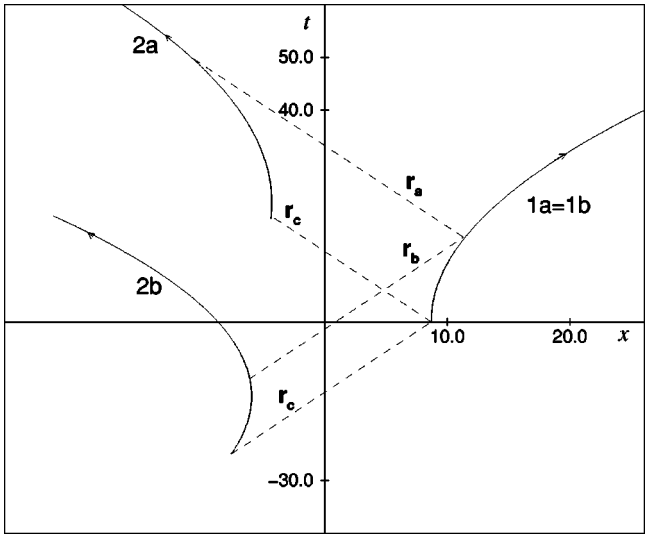

FIG. 3. Matching the trajectory of particle 1 for a generic $F(r)$: Particle 2 starts from the past in case b and, unless $F(r)$ is chosen precisely, its future does not coincide with the trajectory of particle 2 of case a. Arbitrary units with $c = 1$ and $e^2 = mc^2$.

Our numerical method produces two trajectories for particle 2 from each set of k_n, by direct numerical integration of the main equations (45) and (46) accompanied by the driven equations for the trajectories:(dx_{2a}/dr_a), (dt_{2a}/dr_a), (dx_{2b}/dr_b), and (dt_{2b}/dr_b) as determined by Eqs. (31) and (32). We calculate the trajectories numerically by using a 9/8 embedded Runge-Kutta pair. In general, two different trajectories are obtained for particle 2, as illustrated in Fig. 3, and we calculate numerically the average squared deviation of the two trajectories over a grid of positions

$$A(k) = \sqrt{\frac{1}{N}\sum_{i=1}^{N} [t_{2a}(x_{2i}) - t_{2b}(x_{2i})]^2}. \tag{59}$$

After that we implement a steepest-descent search in the 18-dimensional parameter space governed by the quenching equation $dk_n/ds = -\partial A/\partial k_n$ until it finds a minimum value for the squared deviation of Eq. (59) (see Ref. [20] for an analogous numerical quenching procedure).

In Fig. 4 we put the converged trajectory of particle 1 superposed to the reflected trajectory of particle 2, for velocities $v/c = 0.46$, $v/c = 0.54$, and $v/c = 0.71$. Notice that the trajectories coincide perfectly, indicating that the quenching search satisfied all the symmetry relations and thus found a consistent solution. As the solution is self-consistent, we cannot set the asymptotic velocity directly, and we determine a final low velocity by using an initial condition with $r_c \gg 1$, while a large asymptotic velocity is achieved by using $r_c \approx 1$ (one classical electronic radius). We start the numerical work at low velocities, by setting a large value of r_c, which results in a small asymptotic velocity. After that, we

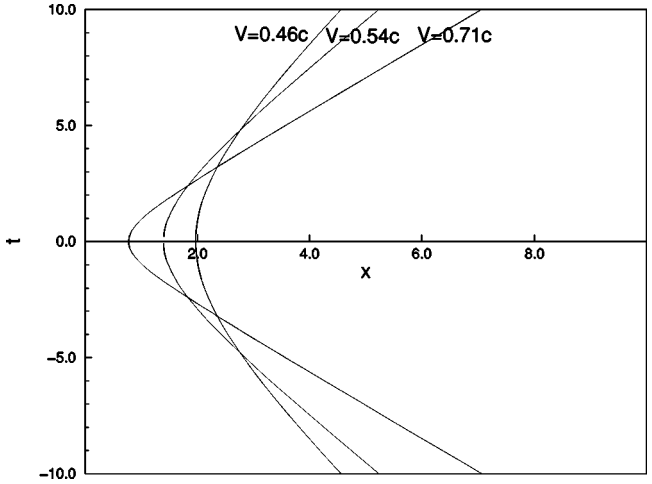

FIG. 4. Numerically determined trajectories in the CMF for $m_1=m_2$ in the (x,t) plane; units with $c=1$ and $e^2=mc^2$. Three different symmetric trajectories found by the steepest-descent method; the orbit of particle 2 is reflected and superposed onto that of particle 1 to show the agreement: $v/c=0.46$, $r_c=4.7$; $v/c=0.54$, $r_c=3.7$; and $v/c=0.71$, $r_c=2.97$.

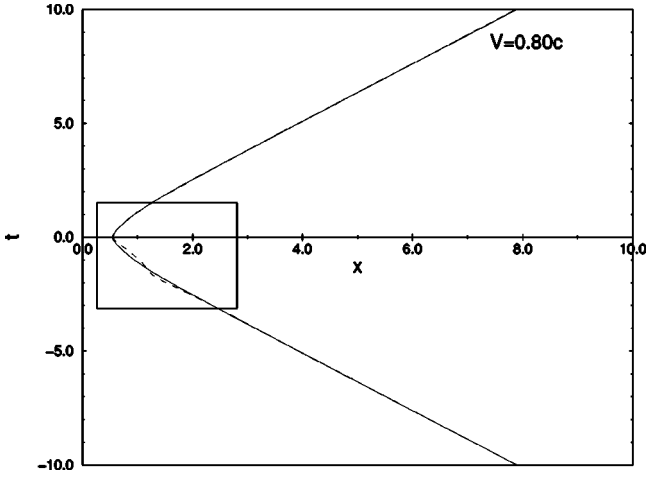

FIG. 5. Numerically determined trajectories in the CMF for $m_1=m_2$ in the (x,t) plane; units with $c=1$ and $e^2=mc^2$. The orbit of particle 2 is reflected and superposed onto that of particle 1 for a highly relativistic case of $v/c=0.8$. Notice the slight mismatch of the two orbits, due to failure of convergence of the series.

decrease the value of r_c and give the formerly determined solution as seed to the quenching method, which converges much faster, as there must be a solution in the neighborhood of the seed.

In Table I we list the final velocities v/c as a function of the initial condition r_c. From the numerically converged trajectories we calculate the minimum distance r_o and Table I shows r_o/r_c. We observe that some coefficients k_n converge to a value below the numerical precision of 10^{-14} such that only the first N coefficients are significant to the numerical precision. This number increases with the asymptotic velocity, as can be seen from Table I.

In Fig. 5 we plot the trajectory of $v/c=0.80$, which took 12 h of numerical quenching to converge and still one can observe a slight mismatch of the orbits in the turning region, indicating the slowness of the convergence process. For this case, the numerical 19th coefficient is still important, indicating that our basis is failing to converge to the solution. This failure suggests that something physical is happening above $v/c=0.71$, maybe we are even loosing the twice-monotonic property (according to Appendix A this could happen at any point above $v/c=0.33$). There might also be another trajectory in the neighborhood, which interferes with the convergence, and last, at high energies Eq. (58) becomes too singular at the collision. Further numerical studies are needed to determine if some special bifurcation is happening to the orbit above $v/c=0.71$, which must be performed with a regularized integration method. In this high-velocity region the functional problem posed on $F(r)$ might not have a unique solution, and for relativistic velocities it is likely that it does not. The symmetric solution was actually proved to be unique only up to a small velocity [12], and we managed to go much above the low limit set by Driver in Ref. [12]. In Fig. 6 we plot $F(r)$ versus (r_o/r) for the asymptotic velocities $v/c=0.46$, $v/c=0.54$, and $v/c=0.80$. Notice that the functional form of $F(r)$ is approximately a linear function of (r_o/r) at low velocities, but at larger velocities it becomes highly convoluted.

TABLE I. Numerically calculated asymptotic velocities v/c, minimal radii r_o/r_c, and number N of significant terms of the $F(r)$ series as a function of the critical initial distance r_c at $\Phi=0$. Units of energy are set by $c=1$ and $e^2=mc^2$.

v/c	r_c	r_0/r_c	N
0.11	60.00	0.9959	5
0.21	22.00	0.9881	6
0.30	10.00	0.9693	7
0.40	6.00	0.9413	9
0.51	4.00	0.8865	13
0.60	3.27	0.8213	16
0.71	2.96	0.7085	17
0.80	2.55	0.7199	18

As a test for the existence of other types of orbits, we integrated Eqs. (45) and (46) using two completely general (and possibly different) functions $F_a(r_a)$ and $F_b(r_b)$, each defined by an independent truncated power series like

$$F_{a,b}(r)=k_o^{a,b}-\sum_{n=1}^{n=9}\frac{k_n^{a,b}}{r^n}. \tag{60}$$

Notice that the saturation value is not anymore $k_o^{a,b}=1$ like in Eq. (58), but rather a generic quenchable value in each case. The integration procedure was started with $r_a\neq r_b$ at $\Phi=0$ for particle 1. We found that the quenching method converged to functions $F_a(r_a)$ and $F_b(r_b)$ that were generally different but were always related by the scaling discussed in Appendix B [see Eq. (B4) and text below it]. According to the discussion of Appendix B this is the case when the orbit is symmetric in another Lorentz frame. This above-

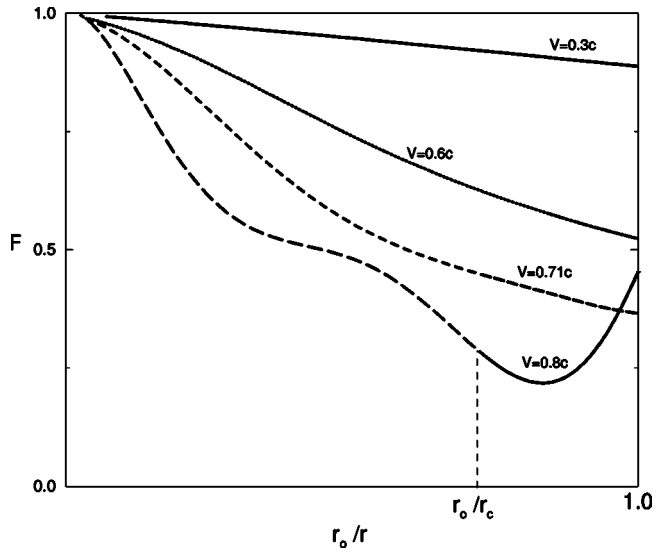

FIG. 6. Numerically determined functions $F(r)$, rescaled as a function of r_o/r; both quantities are dimensionless. Notice that at low velocities $F(r)$ is well approximated by the first two terms of the series, $F(r)\sim 1-k_1/r$, but at larger velocities $F(r)$ becomes highly convoluted.

defined search would be capable of finding any existing orbit with the twice-monotonic property and the fact that it always converged to Lorentz-transformed symmetric orbits is indicative that there are no other types of low-energy solutions.

At this point it is interesting to appreciate the big detour taken by our numerical method to solve Eq. (1), which should be compared to the most straightforward way to solve a neutral-delay differential equation like Eq. (1), namely, postulating an initial function and continuing the solution by use of the differential equation. The straightforward method necessarily leads to runaways because one is never capable of guessing the unique nonrunaway initial function, and even if one does guess the nonrunaway condition, numerical roundoff plagues the integration and one still gets runaways after some short time. Our numerical method is superior in this respect precisely because it is already placed in the nonrunaway manifold, and the quenching implemented to solve the functional problem for $F(r)$ is numerically stable, as it does not involve extrapolation.

VI. CONCLUSIONS AND DISCUSSION

In this paper we discussed the solution and Hamiltonian description of the time-symmetric two-body problem of the action-at-a-distance electrodynamics with repulsive interaction. Our method is closed and does not involve expansions, only the hypothesis that the orbit is twice monotonic was used. We conjecture that our solution is already the general solution at low energies, which can be argued in the following way: For a generic solution, possibly unsymmetrical, one can always find a Lorentz frame where the asymptotic outbound velocities of the two particles are opposite (the outbound CMF). Now *if* asymptotic data determine a unique trajectory for low energies, that trajectory is our symmetric solution in this CMF. Of course the symmetric solution is always a possible solution, but the symmetry of the physics in this outbound CMF suggests that it is the only solution for low energies, as the solution has to correspond to the Coulombian solution, which has this property (this would actually be a nontrivial generalization of the work of Driver and Hoag in Refs. [12,13]). With the above in mind, the Hamiltonian we derive here is already the general order-reduced Hamiltonian for low energies. Some few numerical experiments have suggested that the conjecture is correct at low energies. Even the high-energy solutions found in Ref. [15] exhibit the property that the future of one particle is the past of the other, and it looks like the numerical method in Ref. [15] simply picked a Lorentz frame slightly off the CMF, but that requires further investigation.

The idea to remove the field degrees of freedom goes back to Dirac [7] and later Wheeler and Feynman planned to quantize WF2B as a means to avoid the divergencies of QED, as in the action-at-a-distance theory the infinite number of field degrees of freedom is absent. History says that the famous seminar that never came from Wheeler (see Ref. [21], p. 97) was due to difficulties in converting the Fokker Lagrangian (3) to the Hamiltonian form. This task is still not fully done and in this work we took a step in that direction for the one-dimensional case at low energies. Notice that the implicit dependence of the Hamiltonian operator is actually convenient for an eigenvalue equation, and one could discuss a canonical quantization procedure based on either Eq. (22) or (23), using the numerically determined potentials. Of course Wheeler and Feynman were mainly interested in the attractive case, of greater relevance for atomic physics and specially for the Lamb shift calculation. The attractive problem is being published elsewhere [19]. In this same Chapter 5, p. 97 of Ref. [21], Feynman says that ''I didn't solve it either—a quantum theory of half-advanced half-retarded potentials—and I worked on it for years . . . .'' This is still an outstanding problem today and the difficulties in casting relativistic *Lagrangian* interactions into the *Hamiltonian* form are well explained in Refs. [22,23]. The only studies we know of dealing with the time-symmetric problem involve power expansions. We are aware of another attempt at a Hamiltonian description of 1D-WF2B that ends up with an infinite-dimensional Hamiltonian [24], such that further order reduction is needed to select nonrunaway orbits.

Our description might seem to violate the no-interaction theorem [1,2], but there are two places where we avoid it: (i) the no-interaction theorem is an obstacle to covariant Hamiltonian description of two interacting particles only in fully three-dimensional motion, we are restricted to one-dimensional motion; (ii) the evolution parameter in our Hamiltonians is not time but rather ζ for case *a* or ξ for case *b* and, therefore, the no-interaction theorem does not apply. In principle, because time is not the evolution parameter, even in three dimensions the no-interaction theorem would not be an obstacle to an analogous procedure, and that is an open problem.

As regards the applied mathematics literature of delay, the theory of delayed functional equations [28,29] is a difficult and poorly investigated subject but it turns out that there are already a few results worth noticing: In a paper of 1974 by

Kaplan and Yorke [25], it was noticed that for some special types of delay equations, solutions can be found by searching the *periodic orbits* of an associated *ordinary* differential equation. This was further generalized in 1999 [26] and it was shown that for a large class of delay equations, the associated ODE turned out to be a *Hamiltonian* ODE, quite a curious result [26] brought up by applied mathematicians with no relation to either quantum mechanics or Wheeler-Feynman electrodynamics. Another set of studies of applied mathematics focuses on the similarities of delay equations to either ODE's or extended systems [27]: if the delay is small and bounded, the behavior should be reminiscent of that of ODE's, as determined by the dimension of the attractor in several systems with small delays, a generic class that contains the bound states of the attractive 1D-WF2B, apart from the fact that our system is conservative. In the limit where the delays are very large, delay equations are found to behave like extended systems, with large dimensional attractors, which is the generic class of the repulsive case of 1D-WF2B, where the lags are unbounded, and also of the unbound states of the attractive case.

As we mentioned in the Introduction, the action-at-a-distance electrodynamics is capable of describing the whole of classical electrodynamics as a limiting case, and even better, a limiting case without the complications of mass renormalization, as demonstrated by Wheeler and Feynman [3]. This was actually what led Wheeler and Feynman to the action-at-a-distance electrodynamics in the first place, but in doing that they formulated a very complex *conservative* physical theory (the conserved energy associated with the Fokker Lagrangian is discussed, for example, in Ref. [17]). It is important to stress that the converse of the above statement is not true at all: the complex *conservative* dynamics of the action-at-a-distance theory is not reducible to a limiting case of Maxwell's electrodynamics (which is always a dissipative theory because of the radiation). Relativistic action-at-a-distance shares the conservative character with Newtonian gravitation, and in the presence of a universe of particles, an atom described by the action-at-a-distance theory has the possibility to behave in a way analogous to the solar system in the Newtonian sky: distant solar masses being just small perturbations, as opposed to the description by Maxwell, where it *has* to radiate. It appears to us that the analysis of the complex conservative dynamics of WF2B is bound to reveal interesting physical insights. As we have seen with this special case study, the physical nonrunaway condition performs the magical reduction from the infinite-dimensional dynamical system posed by the delay equation to a finite-dimensional one, and the large body of existing understanding on qualitative behavior of finite-dimensional vector fields should be applicable [28]. Some results already published for systems of atomic physics within the Darwin approximation, a low-velocity Hamiltonian approximation to action-at-a-distance, have already revealed interesting news [30,31]. Existing numerical methods for the relativistic case are still short reaching [32] and cannot reproduce the massive numerical search performed with the Darwin approximation in [31].

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APPENDIX A: PROOF OF THE TWICE-MONOTONIC PROPERTY

In this section we show that in the Coulombian limit of a low-energy orbit, the solution of Eqs. (1) and (2) has only two branches, one defined by $\dot{r}>0$ and another defined by $\dot{r}<0$. Because of time-reversal symmetry, the theorem is the same for either r_a [indicated by q in Eqs. (1) and (2)] or by r_b [indicated by r in Eqs. (1) and (2)]. It suffices to prove that there is only one point where $\dot{q}$ vanishes, with q defined in Eq. (2). A special version of this proof was given in Ref. [12] along symmetric orbits of the equal-mass case. The proof is trivial and can be done for a generic orbit of the arbitrary-mass repulsive two-body system in the CMF: We start from the definition of the light-cone condition for a generic CMF orbit,

$$q=x_1(t)-x_2(t+q), \tag{A1}$$

where $x_1(t)$ represents the position of particle 1, assumed on the right, and $x_2(t)$ represents the position of particle 2, assumed on the left, and we have set $c=1$. Notice that Eq. (2) is a special case of Eq. (A1) for symmetric orbits of the equal-mass case $[x_2(t)=-x_1(t)]$. The key observation is that because the interaction is always repulsive, the velocity $v_1(t)$ of particle 1 is a monotonically *increasing* function of time (particle 1 is repelled to the right), while the velocity $v_2(t)$ of particle 2 is a monotonically *decreasing* function of time (particle 2 is repelled to the left). If we take the derivative of Eq. (A1) with respect to t and isolate $\dot{q}$, we obtain

$$\dot{q}=\frac{v_1(t)-v_2(t+q)}{1+v_2(t+q)}. \tag{A2}$$

For low energy we have the bounds $|v_2(t+q)|<v_2(\infty)\ll 1$ and $|v_1(t)|<v_1(\infty)\ll 1$, and therefore the denominator of Eq. (A2) is always positive. In the CMF, the value of $\dot{q}$ changes sign from the inbound asymptotic region to the outbound asymptotic region, with values in the interval

$$-\left(\frac{v_1(\infty)+v_2(\infty)}{1+v_2(\infty)}\right)<\dot{q}<\left(\frac{v_1(\infty)+v_2(\infty)}{1-v_2(\infty)}\right). \tag{A3}$$

To complete the proof we need only to notice that the sum of two monotonically increasing functions $[v_1(t)$ and $-v_2(t+q)]$ is also monotonically increasing and, therefore,

can only vanish once. It should be noticed that $-v_2(t+q)$ is not necessarily a monotonically increasing function of t for high-velocity orbits, as

$$\frac{dv_2(t+q)}{dt}=(1+\dot{q})\frac{dv_2(t+q)}{d(t+q)}, \tag{A4}$$

which is the product of $1+\dot{q}$ times a negative number [recall that $v_2(t)$ is a monotonically decreasing function of its argument and therefore, $-v_2(t)$ is a monotonically increasing function]. It can be seen that Eq. (A4) guarantees that $v_2(t+q)$ is an increasing function of t if $(1+\dot{q})$ is positive, which is the case for a low-velocity orbit. A simple estimate for the first velocity where the twice-monotonic property can fail in the equal-mass case is given by setting $|\dot{q}|=1$ in Eq. (A3), which predicts $v(\infty)=1/3$.

APPENDIX B: COVARIANT DEFINITION OF THE EQUAL-MASS CASE

In this appendix we exhibit a covariant derivation of the above Hamiltonization procedure, which we develop only for the equal-mass case and with the hypothesis that the orbit is twice monotonic (proved in Ref. [12] for low energies and in Appendix A for arbitrary mass in the CMF). The definition of a covariant family starts from the observation that the Lorentz transformation of a symmetric orbit has the property that the future of particle 1 is the past of particle 2 and vice versa. We define the relativistic symmetric family of orbits, henceforth called RSF, as the family of orbits with the property that *if* $x_1(t_1)+x_2(t_2)=0$, then it follows that $t_1+t_2=0$. It is easy to verify that RSF is a Lorentz invariant family of orbits and also that any Lorentz transformation of a symmetric orbit belongs to RSF (but in principle these should not exhaust the RSF: there could be other types of orbits). A generic orbit of the RSF is represented in Fig. 7, where we illustrate the time-reversal-and-exchange symmetry. The above definition implies that the future of particle 1 is the past of particle 2 in the RSF. Inside the RSF, by use of the time-reversal operation $\zeta_1\rightarrow-\xi_2$, $\zeta_2\rightarrow-\xi_1$, we can prove the same identities (13), (14) and an equivalent form of Eq. (15), relating particle 1 to particle 2 (only that in case b the role of α and β is exchanged), and these in turn lead to the same type of general ghost Lagrangian to describe a twice-monotonic orbit.

The relativistic condition that the future of one particle is the past of the other implies that the solution of WF2B inside the RSF must have the following branch correspondences: $v_{1a}^{t,d}(r_a)=v_{2a}^{d,t}(r_a)$ and $v_{1b}^{t,d}(r_b)=v_{2b}^{d,t}(r_b)$ (see Fig. 7). These conditions can be seen with the help of Eqs. (31) and (32) to be equivalent to the four conditions

$$\frac{\partial p_{a,b}^{t,d}}{\partial P}=-\frac{\partial p_{a,b}^{d,t}}{\partial P}, \tag{B1}$$

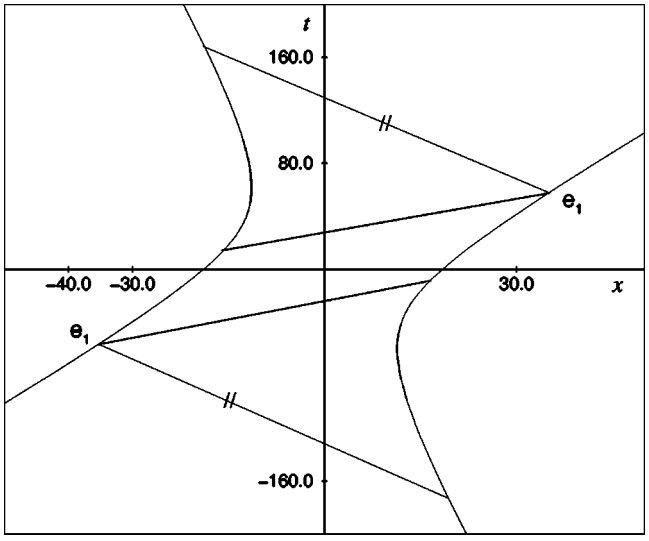

FIG. 7. A typical trajectory of the RSF in a generic Lorentz frame for the equal-mass case in the (x,t) plane; units with $c=1$ and $e^2=mc^2$. The only symmetry is that the future of particle 1 is the past of particle 2.

$$\frac{\partial p_{a,b}^{t,d}}{\partial E}=-\frac{\partial p_{a,b}^{d,t}}{\partial E}.$$

The above conditions imply that the orbit is defined by four different functions $\pm s_a(r)$, $\pm s_b$, $F_a(r)$ and $F_b(r)$ [see definitions (36) and (37)]. To verify that condition (B1) is relativistically invariant, let us suppose that we tried to describe the orbit from another Lorentz frame, with boost parameter w. If the orbit is twice monotonic, it can be described in case a with a Lagrangian $\bar{L}_a$ of the same type as Eq. (20), and a Hamiltonian of type (22), and the new coordinates $\bar{x}_{1a},\bar{t}_{1a},\bar{x}_{1b},\bar{t}_{1b}$ must be obtained by a simple Lorentz transformation with boost parameter w. Imposing this condition on the explicit solution (31), and noticing that the advanced light-cone distance in the new frame is related to the old one by $d\bar{r}_a=\sqrt{(1+w)/(1-w)}dr_a$, we obtain

$$\frac{\partial\bar{p}_a}{\partial E}=\frac{(1-w)}{(1+w)}\frac{\partial p_a}{\partial E}, \tag{B2}$$

$$\frac{\partial\bar{p}_a}{\partial P}=\frac{\partial p_a}{\partial P},$$

which, besides showing that Eq. (B1) is frame independent , also shows that $s_a(r)$ as defined by Eq. (36) is a Lorentz scalar:

$$\bar{s}_a(\bar{r}_a)=s_a(\lambda_a\bar{r}_a), \tag{B3}$$

and that $F_a(r_a)$ [as defined by Eq. (37)] transforms like

$$\overline{F}_a(\overline{r}_a)=\lambda_a^2 F(\lambda_a \overline{r}_a), \tag{B4}$$

with $\lambda_a=\sqrt{(1-w)/(1+w)}$. Case b transforms in the same way with $\lambda_b=1/\lambda_a$. This last equation allows us to express the Hamiltonian in any frame by use of the CMF form of $F(r)$ and a rescaling depending on the boost parameter, which can be determined by asymptotic data.

Last we show that the action of a Lorentz transformation on Hamiltonian (22) is a canonical transformation: It is easy to verify with the help of Eq. (4) that a Lorentz transformation simply rescales the coordinates ξ_1 and ξ_2 to $\overline{\xi}_1=(1/\lambda)\xi_1$ and $\overline{\xi}_2=(1/\lambda)\xi_2$ with $\lambda=\sqrt{(1-w)/(1+w)}$. To complete the change with a canonical transformation, one must scale the momenta with the inverse factor, $\overline{p}_1=\lambda p_1$ and $\overline{p}_2=\lambda p_2$. By this canonical transformation the transformed Hamiltonian is

$$\overline{H}_a=\frac{-\lambda}{4}\left\{\frac{\overline{M}_{1a}^2(\overline{r}_a)}{\left(\overline{p}_1+\frac{1}{2}\overline{V}+\frac{e^2}{|\overline{\xi}_1-\overline{\xi}_2|}\right)}+\frac{\overline{M}_{2a}^2(\overline{r}_a)}{\left(\overline{p}_2+\frac{1}{2}\overline{V}+\frac{e^2}{|\overline{\xi}_1-\overline{\xi}_2|}\right)}\right\}-\lambda\overline{\phi}(\overline{r}_a), \tag{B5}$$

where $\overline{\phi}(\overline{r}_a)=(1/\lambda)\phi(r_a)$, $\overline{V}(\overline{r}_a)=\lambda V(r_a)$, $\overline{M}_{1a}^2(\overline{r}_a)=M_{1a}^2(r_a)$, and $\overline{M}_{2a}^2(\overline{r}_a)=M_{2a}^2(r_a)$. Notice that Hamiltonian $\overline{H}_a$ picked a multiplicative factor of λ and if we also perform a change to the natural evolution parameter $\overline{\zeta}=\lambda\zeta$ of the new Lorentz frame, it compensates exactly for that factor, going back to the form (22), the same form for all Lorentz frames.

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